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ADOMIAN DECOMPOSITION METHOD FOR SOLVING SIMPLE PENDULUM OSCILLATORY PROBLEMS

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ABSTRACT
In this research, Adomian decomposition method (ADM) is presented to find the numerical solution of the equations arising in oscillatory motion of a simple pendulum. For comparative study Haar wavelet method (HWM) is utilized. Numerical examples illustrate the accuracy of the Adomian decomposition method.

KEYWORDS: Simple Pendulum, Adomian Decomposition Method (ADM), Haar Wavelet Method (HWM), Convergence.

1. INTRODUCTION
The linear and nonlinear differential equations with constant coefficients find their most important applications in the study of electrical, mechanical and other linear or nonlinear systems and these equations play a dominant role in unifying the theory of electrical and mechanical oscillatory systems. A heavy particle or a body attached by a light string to a fixed point and oscillating under gravity constitutes a simple pendulum. Consider a particle of mass m attached to the end of a light inextensible rod, with motion taking place in a vertical plane (as shown in Figure 1).

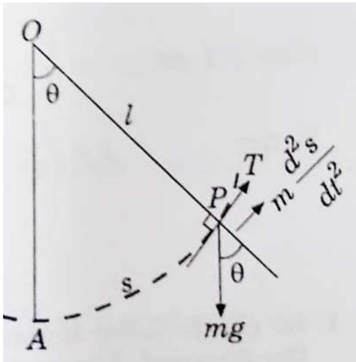

Fig: 1

Let O be the fixed point and l be the length of a string and A be the position of the bob initially. If P be the position of the bob at any time t, such that arc $AP = s$. Let θ be the angle between vertical and line OP, then $s = l\theta$. The equation of motion along PT is:

$$m\frac{d^2s}{dt^2} = -mgsin\theta.$$

After substitution, $s = l\theta$, we obtain

$$\frac{d^2(l\theta)}{dt^2} = -gsin\theta.$$

After simplification, we obtain

$$\frac{d^2\theta}{dt^2} = -\left(\frac{g}{l}\right) sin\theta,$$

$$\frac{d^2\theta}{dt^2} = -\omega^2 sin\theta, \quad \omega = \sqrt{\left(\frac{g}{l}\right)}$$

Decomposition method has developed for solving frontier problems of physics in [1]. Analytic solution of nonlinear boundary-value problems in several dimensions with the help of decomposition method has presented in [2]. Convergence analysis of decomposition methods has been presented in [3]. Adomian decomposition method has presented for solving fourth order integro-differential equations in [4]. Adomian decomposition method for solving second order ordinary differential equations has presented in [5]. Adomian decomposition method and Haar wavelet methods have been presented for solving some oscillatory problems arising in science and engineering in [6]. Adomian decomposition method has been presented for solving nonlinear systems in [7]. Haar wavelet methods have been developed for solving differential and integral equations in [8-12].

2. ADOMIAN DECOMPOSITION METHOD (ADM)

Consider differential equation

$$Ly + Ry + Ny = g(x) \tag{1}$$

where N is a non-linear operator, L is the highest order derivative which is assumed to be invertible and R is a linear differential operator of order less than L.Making Ly subject to formula, we obtain

$$Ly = g(x) - Ry - Ny \tag{2}$$

By solving (2) for Ly, since L is invertible, we can write

$$L^{-1}Ly = L^{-1}g(x) - L^{-1}Ry - L^{-1}Ny \tag{3}$$

For initial value problems we define L^{-1} for $L = \frac{d^n}{dx^n}$ as the definite integration from 0 to x.If L is second-order operator, L^{-1} is integral and by solving (3) for y, we obtain

$$y = A + Bx + L^{-1}g(x) - L^{-1}Ry - L^{-1}Ny \tag{4}$$

where A and B are constants of integration and can be found from the initial or boundary conditions. The Adomian method consists of approximating the solution of (1) as an infinite series.

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{5}$$

and decomposing the non-linear operator N as

$$N(y) = \sum_{n=0}^{\infty} A_n \tag{6}$$

where A_n are Adomian polynomials [19, 20] of $y_0, y_1, y_2, \dots, y_n$ given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty} \lambda^i y_i\right)\right]_{\lambda=0}, n = 0,1,2,\dots$$

Substituting (5) and (6) into (4) we get

$$\sum_{n=0}^{\infty} y_n = A + Bx + L^{-1}g(x) - L^{-1}R\left(\sum_{n=0}^{\infty} y_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right)$$

The recursive relationship is found to be

$$y = g(x)$$

$$y_{n+1} = -L^{-1}Ry_n - L^{-1}A_n$$

Using the above recursive relationship, we can make solution of y as

$$y = \lim_{n\to\infty} \Phi_n(y), \tag{7}$$

where

$$\Phi_n(y) = \sum_{i=0}^{n} y_i \tag{8}$$

3. CONVERGENCE ANALYSIS OF ADM

Consider the equations $y''(t) = f(t, y)$ with $y(0) = y_0,\ y'(0) = y_1$. This equation can be written as:

$$y'' = Ly + N(y), t > 0,\ \ y(0) = f, y'(0) = f'$$

where $L: T \to X$ is a linear operator of form a Banach space T to a Banach space $X(T \subseteq X), N(y): T \to T$ is a nonlinear function on the Banach space T and $f, f^{-1} \in T$ are initial data. Consider the abstract functional equation defined by

$$y = y_0 + y_1(t) + f(y), y \in T$$

where T is a Banach space and $f(y): T \to T$ is analytic near the initial conditions y_0 and y_1.

$$\left.\begin{aligned} Y_n &= y_0 + y_1(t) + \sum_{k=2}^{n} y_k \\ f(Y_n) &= \sum_{k=0}^{n} A_k\,(y_0, y_1, \dots, y_k) \end{aligned}\right\}$$

The ADM is equivalent to determining a sequence $\{Y_n\}_{n\in N}$ from

$$\left.\begin{aligned} Y_0 &= y_0 + y_1(t) \\ Y_{n+1} &= y_0 + y_1(t) + f_n(Y)_n, n \geq 0 \end{aligned}\right\}$$

If the limits

$$\left.\begin{aligned} Y &= \lim_{n\to\infty} Y_n \\ f &= \lim_{n\to\infty} f_n \end{aligned}\right\}$$

exist in Banach space T, then Y solves the fixed point equation $Y = y_0 + y_1(t) + f(y)$ in T .It is also assumed that the following condition holds

$$\left.\begin{aligned} &\|f(y)\|_T \leq 1, \forall\, y \in T \\ &\quad and \\ &\|f_n(Y_n) - f(y)\|_r \to 0\ as\ n \to \infty \end{aligned}\right\}$$

These two conditions are rather restrictive. The first condition implies a constraint on the nonlinear function $f(y)$ and the second condition implies convergence of the series of Adomian polynomial to the locally analytic function $f(y)$.

4. HAAR WAVELETS

Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. Haar wavelet is a sequence of rescaled square shaped functions which together forms a wavelet family or basis. The Haar wavelet function $h_i(x)$ is defined in the interval $[\alpha\ \gamma]$ as:

$$h_i(x) = \begin{cases} 1, & \alpha \leq x < \beta \\ -1, & \beta \leq x < \gamma \\ 0, & elsewhere\,, \end{cases}$$

where $\alpha = \frac{k}{m}, \beta = \frac{k+0.5}{m},\ \gamma = \frac{k+1}{m},\ m = 2^j$ and $j = 0, 1, 2, 3, \dots, J$. J denotes the level of the resolution. The integer $k = 0,1,2, \dots, m-1$ is the translation parameter. The index i is calculated as $i = m + k + 1$. The minimal values $i = 2$ and the maximal value of $i = 2^{j+1}$. The collocation points are calculated as

$$x_l = \frac{(l - 0.5)}{2M},\ \ l = 1,2,3, \dots \dots, 2M.$$

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The operational matrix P, which is $2M \times 2M$, is calculated as below

$$P_{1,i}(x) = \int_0^{x_l} h_i(x)dx$$

$$P_{n+1,i}(x) = \int_0^{x} P_{n,i}(x)dx, \qquad n = 1, 2, 3, \ldots.$$

5. DAMPED AND UNDAMPED SIMPLE PENDULUM OSCILLATORY PROBLEMS

Consider the differential equation of simple pendulum is

$$\frac{d^2\theta}{dt^2} + \omega^2 sin\theta = 0 \tag{9}$$

where θ is the angular displacement, t is the time , $\omega^2 = \frac{g}{l}$ is the natural frequency of the small oscillations of the pendulum, l is the length of the pendulum and g is the acceleration due to gravity.

The oscillations of the pendulum are subjected to the initial conditions:

$$\theta(0) = a, \;\; \theta'(0) = 0 \tag{10}$$

where a represents the amplitude of the oscillations. For small angles, use the series expansion

$$sin\theta \cong \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - - - \tag{11}$$

From (9) and (11), we obtain

$$\theta'' + \omega^2\theta - \frac{1}{6}\omega^2\theta^3 + \frac{1}{120}\omega^2\theta^5 - - - - = 0 \tag{12}$$

The simple pendulum equation with damping force is

$$\theta'' + 2k\theta' + \omega^2\theta - \frac{1}{6}\omega^2\theta^3 + \frac{1}{120}\omega^2\theta^5 - - - - = 0 \tag{13}$$

6. HAAR WAVELET METHOD FOR SOLVING SIMPLE PENDULUM OSCILLATORY PROBLEMS

Consider the approximation

$$\theta''(t) = \sum_{i=1}^{2M} a_i h_i(t)$$

Integrating twice w.r.t t, from 0 to t, we obtain

$$\theta'(t) = \theta'(0) + \sum_{i=1}^{2M} a_i P_{1,i}(t)$$

$$\theta(t) = \theta(0) + t.\theta'(0) + \sum_{i=1}^{2M} a_i P_{2,i}(t) \tag{14}$$

Substituting these values in (13) and discretizing using collocation points, we obtain a system of algebraic equations. Solving the system of algebraic equations, we obtain wavelet coefficients. The numerical solution is obtained by substituting the wavelet coefficients into (14).

7. NUMERICAL OBSERVATIONS

In this section, we present some numerical examples to illustrate the accuracy of the proposed method.

Example 1: The equation of undamped pendulum equation (upto first approximation) is

$$\theta'' + \omega^2\theta = 0 \tag{15}$$

The exact solution in this case is $\theta(t) = coswt$. Letting (14) with $k = 0,\ w = 1$ and initial conditions $\theta(0) = 1$, $\theta'(0) = 0$. Applying Adomian decomposition method, we obtain

$$L^{-1}(\theta) = -L^{-1}(\theta),$$

$$\theta(t) = \theta(0) + \theta'(0) - \int_0^t\int_0^t \theta\, dtdt,$$

$$\theta(t) = 1 - \int_0^t\int_0^t \theta\, dtdt \tag{16}$$

Letting,

$$\theta(t) = \sum_{n=0}^{\infty} \theta_n$$

From (16), we obtain

$$\sum_{n=0}^{\infty} \theta_n = 1 - \int_0^t\int_0^t \sum_{n=0}^{\infty} \theta_n\, dtdt \tag{17}$$

$$\theta_0 + \theta_1 + \theta_2 + \cdots = 1 - \int_0^t\int_0^t [\theta_0 + \theta_1 + \theta_2 + \cdots]dtdt \tag{18}$$

From (18), we obtain

$$\theta_0 = 1$$

$$\theta_1 = -\int_0^t\int_0^t \theta_0 dtdt = -\frac{t^2}{2},$$

$$\theta_2 = -\int_0^t\int_0^t \theta_1 dtdt = \frac{t^4}{24},$$

$$\theta_3 = -\int_0^t\int_0^t \theta_2 dtdt = -\frac{t^6}{720},$$

$$\vdots$$

The solution is:

$$\theta = \theta_0 + \theta_1 + \theta_2 + \theta_3 + \cdots$$

$$\theta(t) = 1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{720} + \cdots = cost$$

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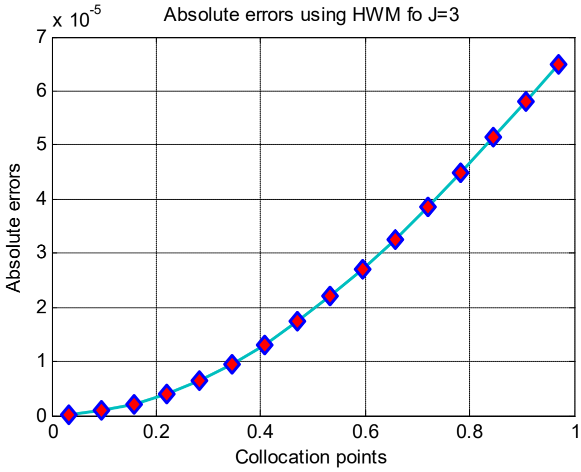

Fig1: Absolute errors obtained in case of HWM for Example 1

Figure 1 shows the absolute error of numerical solutions using Haar wavelet method (HWM) for J=3. Figure 2 shows the comparison of numerical results obtained with ADM (using four terms approximation) and HWM of Example 1.

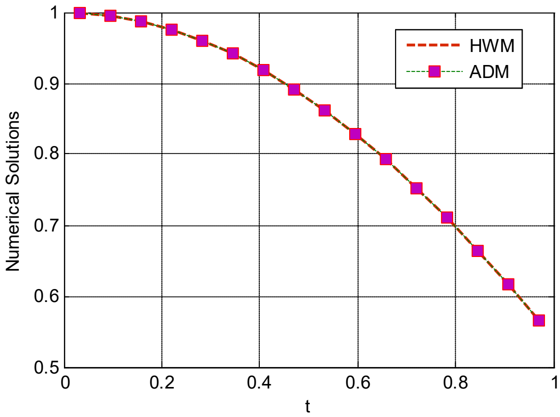

Fig2: Comparison of numerical results of HWM and ADM of Example 1

Example 2: The equation of damped pendulum equation (upto first approximation) is

$$\theta'' + 2k\theta' + \omega^2\theta = 0 \tag{19}$$

The exact solution in this case is $\theta(t) = (1+t)e^{-t}$. Letting (18) with $k = 1$, $w = 1$ and initial conditions $\theta(0) = 1$, $\theta'(0) = 0$. Applying Adomian decomposition method, we obtain

$$L^{-1}(\theta) = -L^{-1}(\theta) - L^{-1}(\theta'),$$

$$\theta(t) = 1 + 2t - 2\int_0^t \theta dt - \int_0^t\int_0^t \theta\, dtdt, \tag{20}$$

Letting,

$$\theta(t) = \sum_{n=0}^{\infty} \theta_n$$

From (20), we obtain

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$$\sum_{n=0}^{\infty}\theta_n = 1+2t-2\int_0^t\sum_{n=0}^{\infty}\theta_n\,dt-\int_0^t\int_0^t\sum_{n=0}^{\infty}\theta_n\,dtdt, \quad (21)$$

$$\theta_0+\theta_1+\theta_2+\cdots = 1+2t-2\int_0^t[\theta_0+\theta_1+\theta_2+\cdots]dt\int_0^t\int_0^t[\theta_0+\theta_1+\theta_2+\cdots]dtdt \quad (22)$$

From (22), we obtain

$$\theta_0 = 1+2t$$

$$\theta_1 = -2\int_0^t\theta_0 dt-\int_0^t\int_0^t\theta_0 dtdt = -2t-\frac{5}{2}t^2-\frac{t^3}{3},$$

$$\theta_2 = -2\int_0^t\theta_1 dt-\int_0^t\int_0^t\theta_1 dtdt = 2t^2+2t^3+\frac{3}{8}t^4+\frac{1}{60}t^5,$$

$$\vdots$$

The solution is:

$$\theta(t) = \theta_0+\theta_1+\theta_2+\cdots = 1-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{8}+\frac{t^5}{30}-\frac{t^6}{144}+\cdots$$

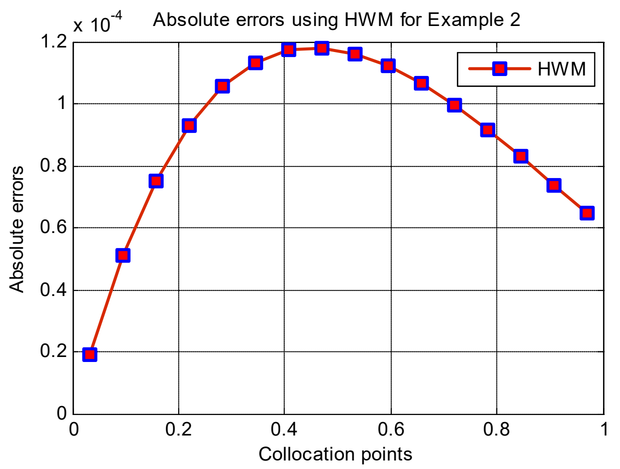

Fig 3: Absolute errors obtained in case of HWM for Example 2

Figure 3 shows the absolute errors of numerical solutions using Haar wavelet method (HWM) for J=3. Figure 4 shows the comparison of numerical results obtained with ADM (using four terms approximation) and HWM of Example 2.

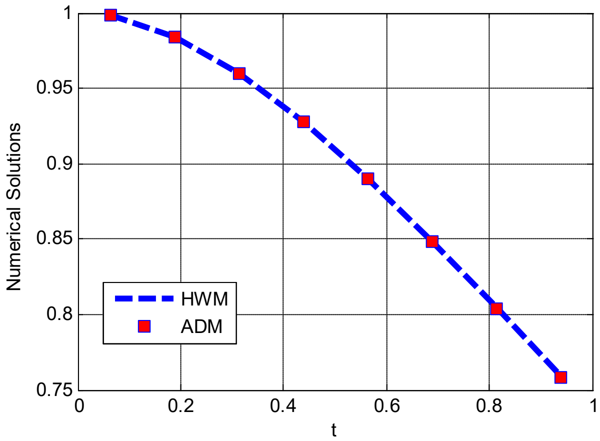

Fig4: Comparison of numerical results of HWM and ADM of Example 2

Example 3: The equation of undamped pendulum equation (upto second approximation) is

$$\theta'' + \omega^2\theta - \frac{1}{6}w^2\theta^3 = 0 \tag{23}$$

Letting (13) with $k = 0,\ w = 1$ and initial conditions $\theta(0) = 1,\ \theta'(0) = 0$. Applying Adomian decomposition method, we obtain

$$\theta(t) = 1 - \int_0^t\int_0^t \theta dtdt + \frac{1}{6}\int_0^t\int_0^t \theta^3 dtdt, \tag{24}$$

Putting,

$$\theta(t) = \sum_{n=0}^{\infty}\theta_n$$

From (24), we obtain

$$\sum_{n=0}^{\infty}\theta_n = 1 - \int_0^t\int_0^t \sum_{n=0}^{\infty}\theta_n\, dtdt + \frac{1}{6}\int_0^t\int_0^t \left[\sum_{n=0}^{\infty}\theta_n\right]^3 dtdt, \tag{25}$$

From (25), we obtain

$$\theta_0 = 1$$

$$\theta_1 = -\int_0^t\int_0^t \theta_0 dtdt + \frac{1}{6}\int_0^t\int_0^t \theta_0^3\, dtdt = -\frac{5t^2}{12},$$

1. $\theta_2 = -\int_0^t\int_0^t \theta_1 dtdt + \frac{1}{6}\int_0^t\int_0^t (\theta_1^3 + 3\theta_0^2\theta_1 + 3\theta_1^2\theta_0)\, dtdt,$

$$\theta_2 = \frac{5}{288}t^4 - \frac{125}{580608}t^8 - \frac{5}{1728}t^6$$

$$\vdots$$

The solution is:

$$\theta = \theta_0 + \theta_1 + \theta_2 + \theta_3 + \cdots$$

$$\theta(t) = 1 - \frac{5t^2}{12}\frac{5}{288}t^4 - \frac{5}{1728}t^6 - \frac{125}{580608}t^8 + \cdots$$

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2. CONCLUSION

From above numerical data, it is concluded that Adomian decomposition method (ADM) and Haar wavelet method (HWM) are powerful mathematical technique for solving simple pendulum oscillation problems arising in many applications of science and engineering. For more accuracy the number terms may be increased.

REFERENCES

[1] G. Adomian, "Solving frontier problem of physics: The Decomposition Method", in Boston: Kluwer Academic Publishers, 1994.
[2] G. Adomian and R. Rach, "Analytic solution of nonlinear boundary-value problems in several dimensions by decomposition" in Journal of Mathematical Analysis and Application, vol. 174(1993): 118-137.
[3] G. Adomian and Y. Cherruault,"Decomposition methods: a new proof of convergence" in Math. Comput. Model, vol. 18, no.12 (1993): 103–106.
[4] I. Hashim, "Adomian decomposition method for solving BVPs for fourth-order integro-differential equations" in Journal of Computational and Applied Mathematics, vol. 193, no. 2 (2006): 658-664.
[5] S. E. Fadugba, S. C. Zelibe and O. H. Edogbanya, "On the Adomian decomposition method for the solution of second order ordinary differential equations" in International Journal of Mathematics and Statistics Studies, vol. 1, no. 2 (2013) :20-29.
[6] I. Haq and I. Singh, "Solving some oscillatory problems using Adomian decomposition method and Haar wavelet method" in Journal of Scientific Research, vol. 12, no. 3 (2020): 289-302.
[7] W. Li and Y. Pang, "Application of Adomian decomposition method to nonlinear systems" in Advances in Difference Equations, vol. 67 (2020): 1-17.
[8] C.F. Chen and C.H. Hsiao, "Haar wavelet method for solving lumped and distributed parameter systems" in IEEE Proceedings: Part D, vol. 144, no. 1 (1997), 87–94.
[9] U. Lepik, "Numerical solution of differential equations using Haar wavelets" in Mathematics and Computers in Simulation, vol. 68 (2005): 127–143.
[10] I.Singh,"Wavelet based method for solving generalized Burgers type equations" in International Journal of Computational Materials Science and Engineering, vol. 8, no. 4 (2019):1–24.
[11] I.Singh and S.Kumar, "Haar wavelet collocation method for solving nonlinear Kuramoto–Sivashinsky equation" in Italian Journal of Pure and Applied Mathematics, vol. 39 (2018): 373-384.
[12] I.Singh and S.Kumar, "Haar wavelet method for some nonlinear Volterra integral equations of the first kind" in Journal of Computational and Applied Mathematics, vol. 292 (2016): 541–552.